

# C4 Exercise 6C (integration using trig. identities)

Note Title

04/07/2007

examples

find  $\int \tan^2 x \, dx$

Since  $\sec^2 x \equiv 1 + \tan^2 x$

then  $\tan^2 x \equiv \sec^2 x - 1$

Hence  $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C$

examples

find  $\int \sin^2 x \, dx$ 

$$\text{Since } \cos 2x \equiv \cos^2 x - \sin^2 x \equiv (-\sin^2 x) - \sin^2 x$$

$$\text{then } \cos 2x \equiv 1 - 2\sin^2 x$$

$$\text{so } 2\sin^2 x \equiv 1 - \cos 2x$$

$$\text{and hence } \sin^2 x \equiv \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\text{then } \int \sin^2 x \, dx = \int \left( \frac{1}{2} - \frac{1}{2}\cos 2x \right) dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

examples

find  $\int \sin 3x \cos 3x \, dx$

now since  $\sin 2\theta = 2 \sin \theta \cos \theta$

we have  $\sin 3x \cos 3x = \frac{1}{2} \sin 2(3x) = \frac{1}{2} \sin 6x$

so  $\int \sin 3x \cos 3x \, dx = \int \frac{1}{2} \sin 6x \, dx = \frac{-1}{12} \cos 6x + C$

examples

find  $\int (\sec x + \tan x)^2 dx$

$$= \int (\sec^2 x + 2\sec x \tan x + \tan^2 x) dx$$

Now  $\int \sec^2 x = \tan x + C$

$$\int 2\sec x \tan x = 2\sec x + C$$

and  $\int \tan^2 x dx = \int (\sec^2 x - 1) dx$  *\* see first example above*  $= \tan x - x + C$

$$\therefore \int (\sec x + \tan x)^2 dx = 2\tan x - x + 2\sec x + C$$

example

find  $\int \sin 3x \cos 2x \, dx$

⚠️ sneaky trick:  $\sin(3x + 2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$

while  $\sin(3x - 2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$

adding get:  $\sin(5x) + \sin(x) = 2 \sin 3x \cos 2x$

so  $\int \sin 3x \cos 2x \, dx = \int \frac{1}{2} (\sin 5x + \sin x) \, dx$

$$= \frac{1}{10} \cos 5x - \frac{1}{2} \cos x + C$$

example:

This clearly will generalise to any expression of the form

$$\sin Ax \cos Bx$$

$$\text{Then } \sin(Ax + Bx) \equiv \sin Ax \cos Bx + \cos Ax \sin Bx$$

$$\text{and } \sin(Ax - Bx) \equiv \sin Ax \cos Bx - \cos Ax \sin Bx$$

$$\Rightarrow 2 \sin Ax \cos Bx \equiv \sin[(A+B)x] + \sin[(A-B)x]$$

$$\Rightarrow \sin Ax \cos Bx \equiv \frac{1}{2} \{ \sin[(A+B)x] + \sin[(A-B)x] \}$$

4Ex6c

1a integrate  $\cot^2 x$

$$\cot^2 x \equiv \operatorname{cosec}^2 x - 1$$

$$\Rightarrow \int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) \, dx$$

$$= -\cot x - x + C$$

4Ex6c

1b integrate  $\cos^2 x$

$$\text{use } \cos 2x \equiv 2\cos^2 x - 1$$

$$\Rightarrow \cos^2 x \equiv \frac{1}{2} (\cos 2x + 1)$$

$$\int \cos^2 x \, dx = \int \left( \frac{1}{2} \cos 2x + \frac{1}{2} \right) dx = \frac{1}{4} \sin 2x + \frac{1}{2} x + C$$



$$\int \sin 2x \cos 2x \, dx = \int \frac{1}{2} (\sin 4x - \sin 0) \, dx \quad \star$$

$\star$  see page 6 of this document

$$\text{since } \sin(2x + 2x) = \sin 2x \cos 2x + \sin 2x \cos 2x$$

$$\Rightarrow \sin 4x = 2 \sin 2x \cos 2x.$$

$$\text{Hence } \int \sin 2x \cos 2x \, dx = \int \frac{1}{2} \sin 4x \, dx$$

$$= -\frac{1}{8} \cos 4x + C$$

1d

$$\int (1 + \sin x)^2 dx = \int (1 + 2\sin x + \sin^2 x) dx$$

$$\cos 2x \equiv \cos^2 x - \sin^2 x \equiv 1 - 2\sin^2 x$$

$$\Rightarrow 2\sin^2 x \equiv 1 - \cos 2x$$

$$\Rightarrow \sin^2 x \equiv \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\Rightarrow \int (1 + \sin x)^2 dx = \int \left( \frac{3}{2} + 2\sin x - \frac{1}{2} \cos 2x \right) dx$$

$$= \frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + C$$

1e

find  $\int \tan^2 3x \, dx$ 

$$\text{Now since } \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \equiv \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$$

$$\begin{aligned} \int \tan^2 3x &\equiv \int (\sec^2 3x - 1) \, dx \\ &= \frac{1}{3} \tan 3x - x + C \end{aligned}$$

11 find  $\int (\cot x - \operatorname{cosec} x)^2 dx$

$$= \int (\cot^2 x - 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x) dx$$

But since  $\frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} \equiv \frac{1}{\sin^2 x} \Rightarrow \cot^2 x + 1 \equiv \operatorname{cosec}^2 x$

$$\begin{aligned} \text{Then } \int (\cot x - \operatorname{cosec} x)^2 dx &\equiv \int (2\operatorname{cosec}^2 x - 1 - 2\cot x \operatorname{cosec} x) dx \\ &= -2\cot x - x + 2\operatorname{cosec} x + C \end{aligned}$$

# CAEx6C

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find  $\int (\sin x + \cos x)^2 dx \equiv \int (\sin^2 x + \cos^2 x + 2\sin x \cos x) dx$

$$\equiv \int 1 + 2\sin x \cos x \, dx$$

Since  $\sin 2x = 2 \sin x \cos x$

$$\int (\sin x + \cos x)^2 dx \equiv \int (1 + \sin 2x) dx$$

$$= x - \frac{1}{2} \cos 2x + C$$

C4 Ex 6C

11h

find  $\int \sin^2 x \cos^2 x \, dx$

$$\sin^2 x \cos^2 x \equiv \cos^2 x - \cos^4 x$$

$$\text{now } \cos^2 2x = (\cos 2x)^2 = (2\cos^2 x - 1)^2$$

$$\Rightarrow \cos^2 2x = 4\cos^4 x - 4\cos^2 x + 1$$

$$\Rightarrow 4(\cos^2 x - \cos^4 x) \equiv 1 - \cos^2 2x$$

$$\Rightarrow \cos^2 x - \cos^4 x \equiv \frac{1}{4} - \frac{1}{4}\cos^2 2x \quad \star$$

$$\text{now } \cos 2\theta \equiv 2\cos^2 \theta - 1 \Rightarrow \cos^2 \theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

replace  $\theta \equiv 2x$  in  $\star$

$$\Rightarrow \cos^2 x - \cos^4 x \equiv \frac{1}{4} - \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)$$

$$\text{So in } \int \sin^2 x \cos^2 x \, dx \equiv \int (\cos^2 x - \cos^4 x) \, dx$$

$$\text{replace this by } \equiv \int \left( \frac{1}{8} - \frac{1}{8} \cos 4x \right) dx$$

$$= \frac{1}{8}x - \frac{1}{32} \sin 4x + C$$

## C4Ex6c

$$1c \quad \int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{1}{(\sin x \cos x)^2} dx$$

$$\text{now } \sin 2x = 2 \sin x \cos x$$

$$\text{so } \sin x \cos x = \frac{1}{2} \sin 2x$$

$$\text{hence } \int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{1}{\frac{1}{4} \sin^2 2x} dx = \int 4 \operatorname{cosec}^2 2x dx$$

$$= -\frac{4}{2} \cot 2x + C$$

$$= -2 \cot 2x + C$$

Or you might use the following method instead...



CH2x6C

i find  $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \operatorname{cosec}^2 x \sec^2 x dx$

$$\frac{\sin^2 x}{\cos^2 x} \equiv \frac{1 - \cos^2 x}{\cos^2 x} \Rightarrow 1 + \tan^2 x \equiv \sec^2 x$$

$$\frac{\cos^2 x}{\sin^2 x} \equiv \frac{1}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x} \Rightarrow 1 + \cot^2 x \equiv \operatorname{cosec}^2 x$$

$$\Rightarrow \int \frac{1}{\sin^2 x \cos^2 x} dx = \int (1 + \cot^2 x)(1 + \tan^2 x) dx$$

$$= \int (1 + \cot^2 x + \tan^2 x + 1) dx$$

$$= \int \{2 + (\operatorname{cosec}^2 x - 1) + (\sec^2 x - 1)\} dx$$

$$\Rightarrow \int \frac{1}{\sin^2 x \cos^2 x} dx = \int (\operatorname{cosec}^2 x + \sec^2 x) dx$$

$$= -\cot x + \tan x + C$$

in the back it gives the solution  $-2 \cot 2x + C$   
are these equal?

$$\cot 2x \equiv \frac{\cos 2x}{\sin 2x} \equiv \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}$$

$$\cot 2x \equiv \frac{1}{2} \cot x - \frac{1}{2} \tan x$$

$$\Rightarrow -2 \cot 2x = -\cot x + \tan x$$

$\therefore$  solution identical.

1j find  $\int (\cos 2x - 1)^2 dx = \int (\cos^2 2x + 1 - 2\cos 2x) dx$

$$\cos 4x \equiv \cos^2 2x - \sin^2 2x$$

$$\cos 4x \equiv 2\cos^2 2x - 1$$

$$\Rightarrow \frac{1}{2} - \frac{1}{2}\cos 4x \equiv \cos^2 2x$$

$$\Rightarrow \int (\cos 2x - 1)^2 dx \equiv \int \left[ \left( \frac{1}{2} - \frac{1}{2}\cos 4x \right) + (1 - 2\cos 2x) \right] dx$$

$$\equiv \int \left[ \frac{3}{2} - \frac{1}{2}\cos 4x - 2\cos 2x \right] dx$$

$$= \frac{3}{2}x - \frac{1}{8}\sin 4x - \sin 2x + C$$

C4Ex6C

2a find  $\int \left( \frac{1 - \sin x}{\cos^2 x} \right) dx \equiv \int (\sec^2 x - \tan x \sec x) dx$

$= \tan x - \sec x + C$

24Ex6C

2b

$$\begin{aligned}\int \frac{1 + \cos x}{\sin^2 x} dx &\equiv \int (\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x) dx \\ &= \cot x + \operatorname{cosec} x + C\end{aligned}$$

C4Ex6c

$$2c \quad \int \frac{\cos 2x}{\cos^2 x} dx$$

$$\cos 2x \equiv 2\cos^2 x - 1$$

$$\text{so} \quad \int \frac{\cos 2x}{\cos^2 x} dx \equiv \int \left( \frac{2\cos^2 x}{\cos^2 x} - \frac{1}{\cos^2 x} \right) dx$$

$$\equiv \int (2 - \sec^2 x) dx$$

$$= 2x - \tan x + C$$

4fx6c

2d

$$\int \frac{\cos^2 x}{\sin^2 x} dx$$

$$\equiv \int \cot^2 x \, dx$$

$$\equiv \int (\operatorname{cosec}^2 x - 1) \, dx$$

$$= -\cot x - x + C$$

2e

$$\int \frac{(1 + \cos x)^2}{\sin^3 x} dx$$

$$\equiv \int \frac{(1 + 2\cos x + \cos^2 x)}{\sin^3 x} dx$$

$$\equiv \int (\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x) dx$$

$$\equiv \int [\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + (\operatorname{cosec}^2 x - 1)] dx$$

$$\equiv \int [2\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x - 1] dx$$

$$= -2\cot x - 2\operatorname{cosec} x - x + C$$



C4fx6c

2f

$$\int \frac{(1 + \sin x)^2}{\cos^2 x} dx$$

$$\equiv \int \frac{(1 + 2\sin x + \sin^2 x)}{\cos^2 x} dx$$

$$\equiv \int \left( \frac{2 - \cos^2 x + 2\sin x}{\cos^2 x} \right) dx$$

$$\equiv \int [2\sec^2 x - 1 + 2\tan x \sec x] dx$$

$$= 2\tan x - x + 2\sec x + C$$

24Ex6C

2g

$$\int (\cot x - \tan x)^2 dx \equiv \int (\cot^2 x - 2 \cot x \tan x + \tan^2 x) dx$$

$$\equiv \int [(\operatorname{cosec}^2 x - 1) - 2 + (\sec^2 x - 1)] dx$$

$$\equiv \int (\operatorname{cosec}^2 x + \sec^2 x - 4) dx$$

$$= -\cot x + \tan x - 4x + C$$

C4Ex6C

2h

$$\int (\cos x - \sin x)^2 dx \equiv \int (\cos^2 x - 2 \sin x \cos x + \sin^2 x) dx \\ \equiv \int (1 - 2 \sin x \cos x) dx$$

$$\text{Now } \sin 2x \equiv 2 \sin x \cos x$$

$$\Rightarrow \int (\cos x - \sin x)^2 dx \equiv \int (1 - \sin 2x) dx \\ = x + \frac{1}{2} \cos 2x + C$$

2i

$$\int (\cos x - \sec x)^2 dx \equiv \int (\cos^2 x - 2\cos x \sec x + \sec^2 x) dx \\ \equiv \int (\cos^2 x + \sec^2 x - 2) dx$$

Now  $\cos 2x \equiv 2\cos^2 x - 1$

$$\Rightarrow \cos^2 x \equiv \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\text{So } \int (\cos x - \sec x)^2 dx \equiv \int \left( \frac{1}{2} \cos 2x + \sec^2 x - \frac{3}{2} \right) dx \\ = \frac{1}{4} \sin 2x + \tan x - \frac{3}{2} x + C$$

C4E×6C

2j

$$\int \frac{\cos 2x}{1 - \cos^2 2x} dx$$

$$\equiv \int \frac{\cos 2x}{\sin^2 2x} dx$$

$$\equiv \int \cot 2x \operatorname{cosec} 2x dx$$

$$\equiv -\frac{1}{2} \operatorname{cosec} 2x + C$$

4Ex 6c

3a

$$\int \cos 2x \cos x \, dx$$

Since:  $\cos(2x + x) \equiv \cos 2x \cos x - \sin 2x \sin x$

And:  $\cos(2x - x) \equiv \cos 2x \cos x + \sin 2x \sin x$

adding we get:  $\cos(3x) + \cos(x) \equiv 2 \cos 2x \cos x$

$$\Rightarrow \cos 2x \cos x \equiv \frac{1}{2} \cos 3x + \frac{1}{2} \cos x$$

$$\begin{aligned} \text{So } \int \cos 2x \cos x \, dx &\equiv \int \left( \frac{1}{2} \cos 3x + \frac{1}{2} \cos x \right) dx \\ &= \frac{1}{6} \sin 3x + \frac{1}{2} \sin x + C \end{aligned}$$

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$$\int 2 \sin 5x \cos 3x \, dx$$

$$\text{Since } \sin(5x + 3x) \equiv \sin 5x \cos 3x + \cos 5x \sin 3x$$

$$\text{and } \sin(5x - 3x) \equiv \sin 5x \cos 3x - \cos 5x \sin 3x$$

$$\text{adding we get } \sin(8x) + \sin(2x) \equiv 2 \sin 5x \cos 3x$$

$$\text{So } \int 2 \sin 5x \cos 3x \, dx \equiv \int (\sin 8x + \sin 2x) dx$$

$$= -\frac{1}{8} \cos 8x - \frac{1}{2} \cos 2x + C$$

3c

$$\int 2 \sin 3x \cos 5x \, dx$$

Since  $\sin(3x + 5x) \equiv \sin 3x \cos 5x + \cos 3x \sin 5x$

and  $\sin(3x - 5x) \equiv \sin 3x \cos 5x - \cos 3x \sin 5x$

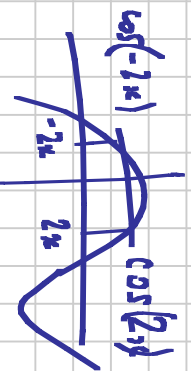
adding we get  $\sin(8x) + \sin(-2x) \equiv 2 \sin 3x \cos 5x$

So  $\int 2 \sin 3x \cos 5x \, dx \equiv \int (\sin(8x) + \sin(-2x)) \, dx$

$$= -\frac{1}{8} \cos(8x) + \frac{1}{2} \cos(-2x) + C$$

you may note that  $\cos(-2x) \equiv \cos(2x)$  to give

$$= -\frac{1}{8} \cos 8x + \frac{1}{2} \cos 2x + C$$



error

here in

earlier

version,

MC.

10.03.14



3d

$$\int 2 \sin 2x \sin 5x \, dx$$

$$\text{Since } \cos(2x - 5x) \equiv \cos 2x \cos 5x + \sin 2x \sin 5x$$

$$\text{and } \cos(2x + 5x) \equiv \cos 2x \cos 5x - \sin 2x \sin 5x$$

subtract the second from the first to derive:

$$\cos(-3x) - \cos(7x) = 2 \sin 2x \sin 5x$$

$$\text{now since } \cos(-3x) \equiv \cos 3x,$$

$$\begin{aligned} \int 2 \sin 2x \sin 5x \, dx &\equiv \int (\cos 3x - \cos 7x) \, dx \\ &= \frac{1}{3} \sin 3x - \frac{1}{7} \sin 7x + C \end{aligned}$$

C4Ex6C

3e

$$4 \int \cos 3x \cos 7x \, dx$$

Since  $\cos(3x + 7x) \equiv \cos 3x \cos 7x - \sin 3x \sin 7x$   
and  $\cos(3x - 7x) \equiv \cos 3x \cos 7x + \sin 3x \sin 7x$

adding, and replacing  $\cos(-4x)$  with  $\cos 4x$

$$\cos 10x + \cos 4x \equiv 2 \cos 3x \cos 7x$$

$$\begin{aligned} \Rightarrow 4 \int \cos 3x \cos 7x \, dx &= 2 \int (\cos 10x + \cos 4x) \, dx \\ &= 2 \left( \frac{1}{10} \sin 10x + \frac{1}{4} \sin 4x \right) + C \\ &= \frac{1}{5} \sin 10x + \frac{1}{2} \sin 4x + C \end{aligned}$$

4Ex6c

3f

$$\int 2 \cos 4x \cos 4x \, dx$$

$$\begin{aligned} \text{now } \cos 2\theta &\equiv \cos^2 \theta - \sin^2 \theta \\ \Rightarrow \cos 2\theta &\equiv 2\cos^2 \theta - 1 \end{aligned}$$

put  $\theta = 4x$  to derive

$$\begin{aligned} \int (2 \cos 4x \cos 4x) \, dx &\equiv \int (1 + \cos 8x) \, dx \\ &= x + \frac{1}{8} \sin 8x + C \end{aligned}$$

Ch 6c

3g

$$\int 2 \cos 4x \sin 4x \, dx$$

$$\sin(4x + 4x) \equiv \sin 4x \cos 4x + \cos 4x \sin 4x$$

$$\Rightarrow 2 \cos 4x \sin 4x \equiv \sin 8x$$

$$\text{So } \int 2 \cos 4x \sin 4x \, dx \equiv \int \sin 8x \, dx$$

$$= -\frac{1}{8} \cos 8x + C$$

CAF6C

3h

$$\int 2 \sin 4x \sin 4x \, dx$$

$$\text{since } \cos 2\theta \equiv \cos^2 \theta - \sin^2 \theta$$

$$\Rightarrow \cos 2\theta \equiv 1 - 2\sin^2 \theta$$

$$\Rightarrow 2\sin^2 \theta \equiv 1 - \cos 2\theta$$

put  $\theta = 4x$  above to get

$$\begin{aligned} \int 2 \sin 4x \sin 4x \, dx &\equiv \int (1 - \cos 8x) \, dx \\ &= x - \frac{1}{8} \sin 8x + C \end{aligned}$$